Arithmetic is useful in the world because it is an example of the diagram:

```
symbols  —— symbolic manipulation ——>  symbols
  |       |                               |
relationship | relationship
  |       |                               |
     v     v                           
world  —— physical operation/relation ——>  world
```
Notation: We write \( \phi_1, \ldots, \phi_k \vdash \psi \) when there exists a natural deduction proof with premises \( \phi_1, \ldots, \phi_k \) and conclusion \( \psi \).

Note: this relation talks only about symbolic manipulation, not the meaning of the formulas. But there is a connection to the relation \( \phi_1, \ldots, \phi_k \models \psi \). ....

We say \( \vdash \) is sound if for all formulas \( \phi_1, \ldots, \phi_k \) and conclusion \( \psi \), if \( \phi_1, \ldots, \phi_k \vdash \psi \) then \( \phi_1, \ldots, \phi_k \models \psi \). (Only valid conclusions can be derived.)

We say \( \vdash \) is complete if for all formulas \( \phi_1, \ldots, \phi_k \) and conclusion \( \psi \), if \( \phi_1, \ldots, \phi_k \models \psi \) then \( \phi_1, \ldots, \phi_k \vdash \psi \). (All valid conclusions can be derived.)

**Theorem:** \( \vdash \) is both sound and complete.
This result can be understood as an example of a commutative diagram:

\[
\begin{array}{ccc}
\phi_1, \ldots, \phi_n & \text{--- natural deduction ---} & \psi \\
| & | & | \\
\text{truth} & \text{truth} & | \\
| & | & v \\
v & v & \\
\text{set of worlds} & \text{--- is a subset of ---} & \text{set of worlds} \\
satisfying \phi_1, \ldots, \phi_n & \text{---} & \text{satisfying } \psi
\end{array}
\]

The proof of this result uses mathematical induction:

Let P be some property of the natural numbers 0, 1, 2, \ldots. If

1. **Base case:** it can be shown that 0 has property P
2. **Inductive Case:** Assuming P is true for \(n\), it can be shown that P is true for \(n + 1\) (for any natural number \(n\))

then all natural numbers satisfy P.
Proof of Soundness

To prove soundness, we take \( P(K) = \) “For all formulas \( \phi_1, \ldots, \phi_n, \psi \), for all natural deduction proofs for \( \phi_1, \ldots, \phi_n \vdash \psi \) containing \( K \) or fewer steps, we have \( \phi_1, \ldots, \phi_n \models \psi \).”

The base case is trivial . . . there are no proofs of 0 steps!

Proof of Soundness: Inductive Step

For the inductive step, we have to show that if \( P(K) \) then \( P(K+1) \), i.e.:

Assuming (Inductive Hypothesis):

For all formulas \( \phi_1, \ldots, \phi_n, \psi \), for all natural deduction proofs for \( \phi_1, \ldots, \phi_n \vdash \psi \) containing \( K \) or fewer steps, we have \( \phi_1, \ldots, \phi_n \models \psi \).”

We have to prove

For all formulas \( \phi_1, \ldots, \phi_n, \psi \), for all natural deduction proofs for \( \phi_1, \ldots, \phi_n \vdash \psi \) containing \( K + 1 \) or fewer steps, we have \( \phi_1, \ldots, \phi_n \models \psi \).
So consider a proof for $\phi_1, \ldots, \phi_n \vdash \psi$ of length $K+1$ (or less).

For all but the last step, if $\alpha$ is the conclusion reached at that step, we get $\phi_1, \ldots, \phi_n \models \alpha$ direct from the induction hypothesis.

Consider the last step. Suppose the rule applied is

$$
\begin{array}{c}
A_1 & A_2 \\
\hline
\end{array}
$$

where $A_1$ and $A_2$ are formulae.

It is enough to show that for all formulas $\phi_1, \ldots, \phi_n$, if (1) $\phi_1, \ldots, \phi_n \models A_1$, and (2) $\phi_1, \ldots, \phi_n \models A_2$, then $\phi_1, \ldots, \phi_n \models A$.

(We know (1) and (2) by the induction hypothesis.)

We have to show this for every proof rule.

**Example:** $\land i$

$$
\begin{array}{c}
A_1 & A_2 \\
\hline
A_1 \land A_2 \\
\end{array}
$$

Suppose (1) $\phi_1, \ldots, \phi_n \models A_1$, and (2) $\phi_1, \ldots, \phi_n \models A_2$.

Then

(1) On every line of the truth table where $\phi_1, \ldots, \phi_n$ are true, $A_1$ is true, and

(2) On every line of the truth table where $\phi_1, \ldots, \phi_n$ are true, $A_2$ is true.

SO (by the truth table for $\land$), on every line of the truth table where $\phi_1, \ldots, \phi_n$ are true, $A_1 \land A_2$ is true.
Rules like $\rightarrow e$ and $\lor e$ are a bit more complicated, we skip the details . . .

Completeness Proof

For completeness, we need to show that if $\phi_1, \ldots, \phi_n \models \psi$ then $\phi_1, \ldots, \phi_n \vdash \psi$.

The proof has three steps:

1. If $\phi_1, \ldots, \phi_n \models \psi$ then $\models \phi_1 \rightarrow (\phi_2 \rightarrow \ldots (\phi_n \rightarrow \psi))$.

2. If $\models \phi_1 \rightarrow (\phi_2 \rightarrow \ldots (\phi_n \rightarrow \psi))$ then $\vdash \phi_1 \rightarrow (\phi_2 \rightarrow \ldots (\phi_n \rightarrow \psi))$.

3. If $\vdash \phi_1 \rightarrow (\phi_2 \rightarrow \ldots (\phi_n \rightarrow \psi))$ then $\phi_1, \ldots, \phi_n \vdash \psi$.

Step 1 is trivial from the truth table for $\rightarrow$.
Completeness: Step 2

For step 2, we prove something slightly more general:

**Proposition 1:** If $\models \phi$ then $\vdash \phi$.

To show this, we first prove:

**Proposition 2:** Let $p_1, \ldots, p_n$ be the propositional atoms (constants) in the formula $\phi$. Consider a line $l$ of the truth table of $\phi$. Define the formulae $\hat{p}_i$ by

- $\hat{p}_i$ is $p_i$ if $p_i$ is true on $l$,
- $\hat{p}_i$ is $\neg p_i$ if $p_i$ is false on $l$.

Then

- If $\phi$ is true on $l$ then $\hat{p}_1, \ldots, \hat{p}_n \vdash \phi$.
- If $\phi$ is false on $l$ then $\hat{p}_1, \ldots, \hat{p}_n \vdash \neg \phi$.

The proof of this is another induction . . .

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**Example**

Consider $\phi = (p \lor q) \rightarrow p$:

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \lor q$</th>
<th>$(p \lor q) \rightarrow p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>

We consider lines 1 and 2 . . .
On line 1, \( p \) and \( q \) are both false, so \( \hat{p} = \neg p \) and \( \hat{q} = \neg q \).

We show \( \neg p, \neg q \vdash \neg (p \lor q) \), then use this to show \( \neg p, \neg q \vdash (p \lor q) \rightarrow p \).

On line 2, \( p \) is false, so \( \hat{p} = \neg p \) and \( q \) is true, so \( \hat{q} = q \).

We show \( \neg p, q \vdash p \lor q \), then use this to show \( \neg p, q \vdash \neg ((p \lor q) \rightarrow p) \).
For line 2:

1 : \neg p \quad \text{Premise}
2 : q \quad \text{Premise}
3 : p \lor q \quad 2, \lor_i

4 : (p \lor q) \rightarrow p \quad \text{Assumption}
5 : p \quad 3, 4, \rightarrow e
6 : \bot \quad 1, 5, \neg \epsilon

7 : \neg ((p \lor q) \rightarrow p) \quad 4 - 5, \neg i

Suppose now that \models \phi, i.e., \phi is true on every line of the truth table. Suppose \phi has just two propositional constants \( p_1 \) and \( p_2 \).

Then we can construct the following natural deduction proof:

(Inside the inner boxes, we use Proposition 2 and the fact that \phi is true on every line of the truth table. E.g., the leftmost inner box corresponds to the fact that \phi holds on the line of the truth table where \( p_1 \) and \( p_2 \) are both true, and we put the proof for \( p_1, p_2 \vdash \phi \) there.)
Completeness: Step 3

Suppose \( \vdash \phi_1 \rightarrow (\phi_2 \rightarrow \ldots (\phi_n \rightarrow \psi) \ldots) \) with proof PP. A proof of the following form establishes \( \phi_1, \ldots, \phi_n \vdash \psi \).

1: \( \phi_1 \)  
2: \( \vdots \)  
3: \( \phi_n \)  
4: \( PP \)

1: \( \phi_1 \rightarrow (\phi_2 \rightarrow \ldots (\phi_n \rightarrow \psi) \ldots) \)
2: \( \vdots \)
3: \( \phi_n \rightarrow \psi \)
4: \( 1, x, \rightarrow e \)
5: \( \vdots \)
6: \( \psi \)  
7: \( n, x + n - 1, \rightarrow e \)